NUMERICAL METHODS - CHEAT SHEET

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1 SOLUTION OF EQUATIONS AND EIGENVALUE PROBLEMS

1.1 Solution of equation

Fixed point iteration: x=g(x) method
Newton’s method

1.2 Solution of linear system

Gaussian elimination method
Gauss-Jordan method
Iterative method
Gauss-Seidel method

1.3 Inverse of a matrix by Gauss Jordan method

1.4 Eigen value of a matrix by power method

1.5 Eigen value of a matrix by Jacobi method for symmetric matrix

2 INTERPOLATION AND APPROXIMATION

Interpolation The process of computing the value of a function inside the given range is called interpolation.

Extrapolation The process of computing the value of a function outside the given range is called extrapolation.

2.1 Newton’s forward and backward difference formulas

\[ E[f(x)] = f(x+h) \]
\[ \Delta[f(x)] = f(x+h) - f(x) \]
\[ E = 1 + \Delta \]

Here \( \Delta \) is called Newton’s forward operator and \( E \) is called shift operator.
Newton’s backward difference operator is $\nabla$

$$\nabla = 1 - E$$

Newton’s forward interpolation formula is

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + ...$$

where $u = (x - x_0) / h$

The above formula is used mainly for interpolating the values of $y$ near the beginning of a set of tabular values.

Newton’s backward interpolation formula is

$$y = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + ...$$

where $u = (x - x_n) / h$

The above formula is used mainly for interpolating the values of $y$ near the end of a set of tabular values.

2.2 Lagrangian Polynomials

Newton’s forward and backward interpolation formulae can be used only when the values of dependent variables are equally spaced; where the values of independent variables are not equally spaced, then we can use following Lagrangian’s Interpolation formula and inverse interpolation formula.

Lagrangian interpolation formula for unequal spaces is

$$y = f(x) = \frac{(x-x_1)(x-x_2)...(x-x_n)}{(x_0-x_1)(x_0-x_2)...(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2)...(x-x_n)}{(x_1-x_2)...(x_1-x_n)} y_1 + \frac{(x-x_0)(x-x_1)...(x-x_n)}{(x_2-x_0)...(x_2-x_n)} y_2 + ... + \frac{(x-x_0)...(x-x_n-1)}{(x_n-x_0)...(x_n-x_{n-1})} y_n$$

Lagrangian inverse interpolation formula for unequal spaces is

$$x = f(y) = \frac{(y-y_1)(y-y_2)...(y-y_n)}{(y_0-y_1)(y_0-y_2)...(y_0-y_n)} x_0 + \frac{(y-y_0)(y-y_2)...(y-y_n)}{(y_1-y_0)(y_1-y_2)...(y_1-y_n)} x_1 + \frac{(y-y_0)(y-y_1)...(y-y_n)}{(y_2-y_0)(y_2-y_1)...(y_2-y_n)} x_2 + ... + \frac{(y-y_0)...(y-y_{n-1})}{(y_n-y_0)...(y_n-y_{n-1})} x_n$$
2.3 Divided differences

If the values of \( x \) are given at unequal intervals then we use divided differences. Divided differences take into consideration the change of the values of the function \( f(x) \) and also the changes in the values of the arguments of \( x \).

Suppose the function \( y = f(x) \) assume the values of \( f(x_0), f(x_1), f(x_2), \ldots, f(x_n) \) respectively, where the intervals \( x_1 - x_0, x_2 - x_1, \ldots, x_n - x_{n-1} \) need not be equal.

First divided difference of \( f(x) \)

\[
\begin{align*}
f(x_0, x_1) &= \frac{f(x_1) - f(x_0)}{x_1 - x_0} \\
\text{similarly,} \\
\end{align*}
\]

Second divided difference is

\[
\begin{align*}
f(x_0, x_1, x_2) &= \frac{f(x_2) - f(x_0)}{x_2 - x_0} \\
\end{align*}
\]

Third divided difference is

\[
\begin{align*}
f(x_0, x_1, x_2, x_3) &= \frac{f(x_3) - f(x_0)}{x_3 - x_0} \\
\end{align*}
\]

The above steps are called divided differences of order 1, 2, 3 respectively.

**Property:** The divided differences are symmetrical in all the arguments, i.e. the value of any difference is independent of order of arguments.

**Newton’s divided difference formula** is

\[
f(x) = f(x_0) + (x - x_0)f(x_0, x_1) + (x - x_0)(x - x_1)f(x_0, x_1, x_2) + \ldots
\]

\[
+ (x - x_0)(x - x_1)(x - x_{n-1})f(x_0, x_1, \ldots, x_n)
\]

2.4 Interpolating with a cubic spline

Suppose \( n+1 \) data points are given we have to find the value of \( y \) corresponding to \( x \) where \( x_i < x < x_{i+1} \) where \( i = 0, 1, 2, \ldots n+1 \). In this case we have to use cubic spline method.

1. \( f(x) \) is a cubic polynomial
2. \( f'(x) \) is second degree polynomial.
3. $f'(x)$ is a first degree polynomial.
4. $f(x)$, $f'(x)$, $f''(x)$ are continuous at each point $(x_i, y_i)$ where $i = 0, 1, 2, \ldots n$.

If the above all conditions are satisfied, then $f(x)$ follows cubic spline.

cubic spline formula The cubic spline in $x_{i-1} \prec x \prec x_i$ is

$$f(x) = \frac{1}{6h} \left[ (x - x_i)^3 M_{i-1} + (x - x_{i-1})^3 M_{i} \right] + \frac{x - x_{i-1}}{h} \left[ y_{i-1} - \frac{h^2 M_{i-1}}{6} \right] + \frac{x - x_i}{h} \left[ y_i - \frac{h^2 M_i}{6} \right]$$

where $h = x_i - x_{i-1}$ for all $i$

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2} \left[ y_{i-1} - 2y_i + y_{i+1} \right] \forall i$$

here $i = 1, 2, 3, \ldots n-1$

$M_0 = 0$

$M_n = 0$

$f''(x_i) = M_i$

# Numerical Differentiation and Integration

Numerical differentiation is a process of calculating the derivative of the given function by means of a table of given values of that function. In this process, we have to calculate $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ at a given point is called numerical differentiation. In this case, we have to use Newton’s forward and backward difference formula to compute the derivatives.

In the previous units, we are finding the polynomial curves $y = f(x)$ passing through $n+1$ points. But in this unit we have to calculate the derivative of such curves at a particular point say $X_k$.

If the values of X are equally spaced, we get the interpolating polynomial using Newton’s formula.

If the derivative required at a point nearer to starting value in table, use newton’s forward method and for value at end of the table use newton’s backward method.

In the case of unequal intervals, we can use, Newton’s divided difference formula or Lagrangian interpolation formula.

## 3.1 Differentiation using interpolation formulae

Newton’s forward difference formula is

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2} \Delta^2 y_0 + \frac{u(u-1)(u-2)}{6} \Delta^3 y_0 + \ldots$$
where \( u = (x - x_0) / h \)

\[
\left( \frac{dy}{dx} \right) at (x = x_0) = \frac{1}{h} \left[ \Delta y_0 - \frac{1}{2} \Delta^2 y_0 + \frac{1}{3} \Delta^3 y_0 - \frac{1}{4} \Delta^4 y_0 + \ldots \right]
\]

\[
\left( \frac{d^2 y}{dx^2} \right) at (x = x_0) = \frac{1}{h^2} \left[ \Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12} \Delta^4 y_0 - \frac{13}{16} \Delta^5 y_0 + \ldots \right]
\]

\[
\left( \frac{d^3 y}{dx^3} \right) at (x = x_0) = \frac{1}{h^3} \left[ \Delta^3 y_0 - \frac{3}{2} \Delta^4 y_0 + \ldots \right]
\]

Newton’s backward difference formula is

\[
y = y_n + u \nabla y_n + \frac{u(u+1)}{2!} \nabla^2 y_n + \frac{u(u+1)(u+2)}{3!} \nabla^3 y_n + \ldots
\]

where \( u = (x - x_n) / h \)

\[
\left( \frac{dy}{dx} \right) at (x = x_n) = \frac{1}{h} \left[ \nabla y_n + \frac{1}{2} \nabla^2 y_n + \frac{1}{4} \nabla^3 y_n + \frac{1}{6} \nabla^4 y_n + \ldots \right]
\]

\[
\left( \frac{d^2 y}{dx^2} \right) at (x = x_n) = \frac{1}{h^2} \left[ \nabla^2 y_n + \nabla^3 y_n + \frac{11}{12} \nabla^4 y_n + \ldots \right]
\]

\[
\left( \frac{d^3 y}{dx^3} \right) at (x = x_n) = \frac{1}{h^3} \left[ \frac{3}{2} \nabla^4 y_n + \ldots \right]
\]

### 3.2 Numerical integration by trapezoidal and Simpson 1/3 and 3/8 rules

The term numerical integration is the numerical evaluation of a definite integral say

\[
\int_a^b f(x) \, dx
\]

where \( a \) and \( b \) are given constants and \( f(x) \) be a given function.

When we apply numerical integration to any function of two independent variables, using double integration is called mechanical cubature.

**Trapezoidal rule** The following rule is said to be trapezoidal rule.

\[
I = \frac{h}{2} \left[ (y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1}) \right]
\]

h is called length of the interval.

\[
h = \frac{UL - LL}{no.\ of\ intervals}
\]

**Simpson’s 1/3 rule** is

\[
I = \frac{h}{3} \left[ (y_0 + y_n) + 2(y_2 + y_4 + \ldots) + 4(y_1 + y_3 + \ldots) \right]
\]

Note: In simpson’s 1/3 rule, \( y(x) \) is a polynomial of degree 2. To apply this rule the number of intervals should be even or the number of ordinates must be odd. In simpson’s 1/3 rule, the truncation error is of the order \( h^4 \), i.e.

\[
|E| \leq \frac{(b-a) h^4 M}{180}
\]

where \( M = \max(y_{iv}^0, y_{iv}^1, y_{iv}^2, \ldots) \)
Simpson’s 3/8 rule is

\[ I = \frac{3h}{8} \left[ (y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \ldots + y_{n-3}) + 2(y_3 + y_6 + \ldots) \right] \]

Note: In simpson’s 3/8 rule, \( y(x) \) is a polynomial of degree 3. To apply this rule the number of intervals must be a multiple of 3.

3.3 Romberg’s method

Let \( I = \int_{a}^{b} f(x) \, dx \)

Using trapezoidal or simpson’s rule, let us find the value of the given definite integral by taking the different values of \( h \) or \( \Delta x \).

Let us apply trapezoidal rule for several times by the values as \( h \), \( h/2 \), \( h/4 \), \( h/8 \), \( h/16 \)...

In this type of problems, let

\[ I = I_2 + \frac{I_3 - I_2}{3} \]

where \( I_1 = \) the value of the definite integral, dividing \( h \) into two parts.

Similarly \( I_2 = \) the value of the definite integral, dividing \( h \) into four parts.

Similarly we can calculate the values of

\[ I = I_3 + \frac{I_4 - I_3}{3} \]

This method can continue till we get two successive values are equal.

3.4 Two and Three point Gaussian quadrature formulae

Two point formula In this case change the given interval \( a \) to \( b \) into -1 to 1 using the following procedure. Transformation formula is

\[ x = \left( \frac{a+b}{2} \right) + \left( \frac{b-a}{2} \right) t \]

where \( b \) is the upper limit and \( a \) is the lower limit.

\[ \int_{-1}^{1} f(t) \, dt = f\left(1/\sqrt{3}\right) + f\left(-1/\sqrt{3}\right) \]

Three point formula In this case change the given interval \( a \) to \( b \) into -1 to 1 using the following procedure. Transformation formula is

\[ x = \left( \frac{a+b}{2} \right) + \left( \frac{b-a}{2} \right) t \]
where \( b \) is the upper limit and \( a \) is the lower limit.

\[
\int_{-1}^{1} f(t) \, dt = A_1 f(t_1) + A_2 f(t_2) + A_3 f(t_3)
\]

where \( A_1 = A_3 = 0.5555 \)
\( A_2 = 0.88888 \)
\( t_1 = -0.7745 \)
\( t_2 = 0 \)
\( t_3 = +0.7745 \)

3.5 Double integral using trapezoidal and Simpson’s rules

**trapezoidal rule** is

\[
I = \frac{h}{2} \left[ \text{sum of the four corner values (circle)} + 2(\text{sum of values available in rectangular box}) + 4(\text{sum of remaining values}) \right]
\]

**simpson rule** is

\[
I = \frac{h}{2} \left[ \text{sum of the four corner values (circle)} + 4(\text{sum of values available in rectangular box}) + 16(\text{sum of remaining values}) \right]
\]

4 INITIAL VALUE PROBLEMS FOR ODE

There are several methods for solving differential equations numerically.

1. Taylor’s series method
2. Euler’s method
3. Runge-kutta method
4. Milne’s predictor and corrector method
5. Adam’s predictor and corrector method

4.1 Single step methods

**Taylor’s series method** Taylor’s series about \( x = x_0 \) is given by

\[
y(x) = y_0 + (x - x_0) y_0' + \frac{(x-x_0)^2}{2!} y_0'' + \frac{(x-x_0)^3}{3!} y_0''' + \frac{(x-x_0)^4}{4!} y_0''''
\]

where \( n_0, y_0 \) denotes the initial values of \( x \) and \( y \)

**Euler method for first order equation** eulers method formula or eulers algorithm.

\[
y_{n+1} = y_n + h f(x_n, y_n) \text{ where } n=0,1,2...
\]

modified eulers method formula :

\[
y_{n+1} = y_n + h f\left[x_n + \frac{h}{2}; y_n + \frac{h}{2} f(x_n, y_n)\right]
\]
Fourth order Runge-Kutta method for solving first and second order equations in this method, let us assume that 

\[ \frac{dy}{dx} = f(x, y) \]

be given
diff eqn, the above eqn can be solved under the condition

\[ y(x_0) = y_0 \]

let the increment \( h = \) length of the interval between two values

fourth order ranji kutta method

\[ K_1 = h f(x_0, y_0) \]
\[ K_2 = h f(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}) \]
\[ K_3 = h f(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}) \]
\[ K_4 = h f(x_0 + h, y_0 + k_3) \]

\[ \Delta y = \frac{(K_1 + 2K_2 + 2K_3 + K_4)}{6} \]
\[ y_1 = y_0 + \Delta y \]
\[ x_1 = x_0 + h \]

The increments in \( y \) for the second interval is computed in a similar manner by the following formulae.

\[ K_1 = h f(x_1, y_1) \]
\[ K_2 = h f(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}) \]
\[ K_3 = h f(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}) \]
\[ K_4 = h f(x_1 + h, y_1 + k_3) \]

\[ \Delta y = \frac{(K_1 + 2K_2 + 2K_3 + K_4)}{6} \]
\[ y_2 = y_1 + \Delta y \]
\[ x_2 = x_1 + h \]

4.2 Multistep methods

Milne’s predictor and corrector methods

1. To use Milne’s predictor and corrector formula, we need atleast 4 values.
2. If the initial values are not given, we can obtain these values either by using Taylor’s series method or by R-K method.

3. We can apply milne’s corrector formula only after applying milne’s predictor formula.

Milne’s predictor formula is
\[ y_{n+1, p} = y_{n-3} + \frac{4}{3} \left( 2y'_{n-2} - y'_{n-1} + 2y'_n \right) \]

Milne’s corrector formula is
\[ y_{n+1, c} = y_{n-1} + h \left( y'_{n-1} + 4y'_n + y'_{n+1} \right) \]

Adam’s predictor and corrector methods
Adam’s predictor formula is
\[ y_{n+1, p} = y_n + \frac{h}{24} \left( 55y'_n - 59y'_{n-1} + 37y'_{n-2} - 9y'_{n-3} \right) \]

Adam’s corrector formula is
\[ y_{n+1, c} = y_n + \frac{h}{24} \left( 9y'_{n+1} + 19y'_n - 5y'_{n-1} + y'_{n-2} \right) \]

5. BOUNDARY VALUE PROBLEMS IN ODE AND PDE
to be included later.